arXiv:quant-ph/0602101v3 16 Nov 2006

Irreducible second order SUSY transformations between real and complex potentials

Boris F. Samsonov

Department of Physics, Tomsk State University, 36 Lenin Avenue, 634050 Tomsk, Russia

Abstract

Second order SUSY transformations between real and complex potentials for three important from physical point of view Sturm-Liouville problems, namely, problems with the Dirichlet boundary conditions for a finite interval, for a half axis and for the whole real line are analyzed. For every problem conditions on transformation functions are formulated when transformations are irreducible.

1 Introduction

Non-Hermitian operators started to attract attention of physicists soon after the main foundation of quantum theory was built (see e.g. [1, 2]). A recent numerical observation [3] that some non-Hermitian one-dimensional Hamiltonians may have purely real spectrum re-initiated an attempt to generalize quantum mechanics by accepting non-Hermitian operators for describing physical observables [4] (so called 'complex quantum mechanics', for recent developments see e.g. [5, 6]). In this respect it is worthy of special mention the paper [7] where the authors establish a general criterion for a set of non-Hermitian operators (so called quasi-Hermitian) to constitute a consistent quantum mechanical system with a normal quantum mechanical interpretation.

Another class of non-Hermitian operators, called pseudo-Hermitian, was introduced by Dirac and Pauli and later used by Wick and Lee [1] to overcome some difficulties related with using Hilbert spaces with an indefinite metric. Recently due to Solombrino and Scolarici [8] this concept found a further generalization as weak pseudo-Hermiticity. A recent observation [9] that the real character of the spectrum of a pseudo-Hermitian Hamiltonian h is related to the existence of a pseudo-canonical transformation, which makes h similar to a Hermitian operator, permits us to suppose that there exists a certain overlap between these two classes of non-Hermitian operators, appearing to be the most appropriate candidates for describing physical observables.

From the general point of view if some non-Hermitian operators with a purely real spectrum are similar to Hermitian ones their incorporation into quantum mechanics cannot be considered as a more general approach with respect to the conventional quantum mechanical description. Their use may give (or may not give, for a recent discussion see [10]) only calculational advantages [5, 7]. From the first glance this observation leads to the negative answer to the question whether or not complex quantum mechanics is an extension of the conventional quantum mechanics. But if we take into account the fact that between non-Hermitian operators with a purely real spectrum there

exist such operators which never can be similar to Hermitian ones the above question seems to be still open. In particular, a non-diagonalizable operator has an incomplete system of eigenfunctions and therefore it can never be similar to a Hermitian operator, the eigenfunctions of which form a complete basis in corresponding Hilbert space. Other operators which should be studied in this respect are the ones having spectral singularities in the continuous part of the spectrum, the feature which never appears in the Hermitian case [11].

In [12] an overlap between $\mathcal{PT}$-symmetric quantum mechanics (by this term some authors mean a complex extension of quantum mechanics) and supersymmetric quantum mechanics (SUSY QM, for recent reviews see [13]) was noticed. Furthermore, as it was shown SUSY QM may be useful to transform non-diagonalizable Hamiltonians into diagonalizable forms [14] and remove spectral singularities from their continuous spectrum [15]. These results permit us to hope that SUSY QM may be a very useful tool in complex quantum mechanics.

In this letter we are using the term *SUSY transformations* in its narrow sense as differential transformations between two (exactly solvable) Hamiltonians having almost the same (up to a finite number of levels) spectra and do not discuss underlying algebraic constructions.

Supersymmetric transformations involving first order intertwining operators between one-dimensional Hamiltonians

$$h_{0,1} = -\partial_x^2 + V_{0,1}(x)\,, \qquad \partial_x \equiv d/dx\,, \qquad x \in (a,b) \tag{1}$$

with the possibility for the potentials $V_{0,1}(x)$ to be complex-valued functions were studied in [16, 17]. A succession of two SUSY transformations with equal factorization constants (confluent transformations, see e.g. [18]) was used in [19] to obtain bound states embedded into continuum of scattering states of a complex potential. On the other hand it is clear that if the intermediate Hamiltonian $\widetilde{h}$ of a chain of two first order transformations $h_0 \to \widetilde{h} \to h_1$ has a physical meaning there are no special needs to study the second order transformation leading from h_0 to h_1 directly (2-SUSY transformation); all properties of the Hamiltonian h_1 can be understood at the level of the first order transformation $\widetilde{h} \to h_1$ (1-SUSY transformation). This is not the case if $\widetilde{h} = -\partial_x^2 + \widetilde{V}(x)$ is not a well defined Hamiltonian acting in the same Hilbert space as h_0 and h_1. In this respect for the case when both $V_0(x)$ and $V_1(x)$ are real in [20] the notion of reducible and irreducible SUSY transformations was introduced. The chain is called reducible if $\widetilde{V}(x)$ is real [20] and irreducible otherwise. Evidently as far as complex potentials are concerned chains irreducible in this sense become reducible [16]. Later [21] (for a recent discussion see [22]) another type of irreducible transformations was described. They appear when the potential $\widetilde{V}(x)$ has singularities inside the interval (a,b) where the potentials $V_0(x)$ and $V_1(x)$ are regular. Recently the third possibility for irreducible chains was noticed [23]. It appears if the intermediate potential $\widetilde{V}(x)$ is regular inside (a,b) but the spectrum of $\widetilde{h}$ is completely different of the spectrum of h_0 and no SUSY transformations between their eigenfunctions exist whereas h_0 and h_1 are (almost) isospectral and their eigenfunctions are connected with the help of a second order SUSY transformation. For these two kinds of irreducible chains spectral properties of h_1 cannot be derived from spectral properties of $\widetilde{h}$ even in the case of complex potentials and one needs to analyze second order SUSY transformations between h_0 and h_1 without involving the intermediate potential $\widetilde{V}(x)$. In this Letter we formulate conditions for 2-SUSY transformations between a real potential V_0 and a complex potential V_1 with a purely real spectrum to be irreducible.

2 First and second order SUSY transformations

In this section we briefly review the main properties of first and second order SUSY transformations (for details see e.g. [13, 18, 24]) we need below.

We consider second order ordinary differential equations (Schrödinger equations)

$$(h - E) f_E(x) = 0 \qquad x \in (a, b) \tag{2}$$

with (Hamiltonian) $h = h_0, \widetilde{h}, h_1$ and $f_E(x) = \psi_E(x), \widetilde{\psi}_E(x), \varphi_E(x)$ respectively; $E \in \mathbb{C}$ is a parameter and a, b may be both finite and infinite.

We say that the Hamiltonians h_0 and $\widetilde{h}$ are 1-SUSY partners if there exists a first order differential operator $\widetilde{L}_1$ intertwining h_0 and $\widetilde{h}$, $\widetilde{L}_1 h_0 = \widetilde{h}\widetilde{L}_1$. Similarly, $\widetilde{h}$ and h_1 are 1-SUSY partners if there exists a first order differential operator L_1 such that $L_1\widetilde{h} = h_1 L_1$. Evidently, the second order differential operator $L = L_1\widetilde{L}_1$ intertwines h_0 and h_1

$$L h_0 = h_1 L \tag{3}$$

and h_1 is called 2-SUSY partner for h_0. Once the existence of L is established solutions of equation (2) with $h = h_1$ can be found by applying L to solutions of the same equation with $h = h_0$, $\varphi_E = L\psi_E$, $\psi_E \notin \ker L$. Evidently, similar property takes place for solutions $\widetilde{\psi}_E$ of the Schrödinger equation with the intermediate Hamiltonian $\widetilde{h}$; they are expressed in terms of solutions of the initial equation ψ_E and $u_1(x)$, $h_0 u_1(x) = \alpha_1 u_1(x)$, $\alpha_1 \in \mathbb{C}$ (u_1 is called transformation function and α_1 is factorization constant)

$$\widetilde{\psi}_E(x) = \widetilde{L}_1\psi_E(x) = -\psi'_E(x) + w(x)\psi_E(x) \qquad w(x) = u'_1(x)/u_1(x) \qquad E \neq \alpha_1 \tag{4}$$

$$\widetilde{\psi}_{\alpha_1}(x) = \frac{1}{u_1(x)}\,.$$

The potential $\widetilde{V}(x)$ is expressed in terms of the function $w(x)$ as follows:

$$\widetilde{V}(x) = V_0(x) - 2w'(x)\,. \tag{5}$$

For the next transformation step $\widetilde{h} \to h_1$ the same formulas (4) and (5) with the evident modifications should be used with the only difference that now one distinguishes the confluent case, when the factorization constant α_2 at the second step of transformation coincides with that of the first step, $\alpha_2 = \alpha_1$, from the usual (non-confluent) case when these constants are different $a_2 \neq \alpha_1$.

Using the second order transformation operator $L = L_1\widetilde{L}_1$ one can avoid the intermediate step and go from h_0 to h_1 directly

$$V_1 = V_0 - 2\left[\log W(u_1, u_2)\right]'' \tag{6}$$

$$\varphi_E = L\psi_E = W(u_1, u_2, \psi_E)/W(u_1, u_2)\,. \tag{7}$$

Here and in the following the symbol W with arguments being functions denotes Wronskians, u_1, u_2 and ψ_E are solutions to equation (2) with $h = h_0$ corresponding to the eigenvalues α_1, α_2 (factorization constants), $h_0 u_{1,2} = \alpha_{1,2} u_{1,2}$, and E respectively. Expressions (6) and (7) are known as particular cases of Crum-Krein formulas [25].

Formula (7) defines the operator L for any sufficiently smooth function ψ_E but if ψ_E is a solution to equation (2) other forms of this equation are useful

$$\varphi_E = (E-\alpha_2)\psi_E + (\alpha_1-\alpha_2)\frac{W(u_2,\psi_E)}{W(u_1,u_2)}\, u_1 \tag{8}$$

$$= (E-\alpha_1)\psi_E + (\alpha_1-\alpha_2)\frac{W(u_1,\psi_E)}{W(u_1,u_2)}\, u_2\,. \tag{9}$$

Here the use of equation (2) has been made to express the second derivatives of the functions u_1, u_2 and ψ_E in terms of the functions themselves. Operator L as given in (8-9) maps any two-dimensional space of solutions of equation (2) with $h = h_0$ at $E \neq \alpha_1, \alpha_2$ onto corresponding space of solutions of the same equation with $h = h_1$. The two-dimensional space span(u_1, u_2) is the kernel of L, $Lu_{1,2} = 0$. Despite that with the help of L one can find solutions of the transformed equation corresponding to $E = \alpha_1, \alpha_2$. For this purpose one has to act with L on functions $v_{1,2} \neq u_{1,2}$, $h_0 v_{1,2} = \alpha_{1,2} v_{1,2}$. Using the fact that $W(u_{1,2}, v_{1,2}) = \text{const}$ and putting $\psi_E = v_{1,2}$, $E = \alpha_{1,2}$ in (8) and (9) one readily gets

$$\varphi_{\alpha_{1,2}} = \frac{u_{2,1}}{W(u_1,u_2)} \qquad h_1\varphi_{\alpha_{1,2}} = \alpha_{1,2}\varphi_{\alpha_{1,2}} \tag{10}$$

where we have omitted an inessential constant factor. It is worth to note that the use of these functions for the next step of the second order transformation gives back the initial Hamiltonian h_0 and, hence, the procedure is completely reversible. Our last comment here is that as it follows from (6) to obtain nonsingular for $x \in (a, b)$ potential differences it is necessary that $W(u_1, u_2) \neq 0$ $\forall x \in (a, b)$ which will be supposed to be the case.

For equal factorization constants ($\alpha_1 = \alpha_2 = \alpha$, the confluent case) the function $W(u_1, u_2)$ in (6) should be replaced by

$$W_c(x) = c + \int_{x_0}^{x} u^2(y)dy\,. \tag{11}$$

Constants c and x_0 should be chosen such that $W_c(x) \neq 0$ $\forall x \in (a, b)$. Solution of the Schrödinger equation with the Hamiltonian h_1 are given by

$$\varphi_E(x) = L\psi_E = (\alpha - E)\,\psi_E(x) + \frac{W(\psi_E, u)}{W_c(x)}\, u(x)\,. \tag{12}$$

This formula gives a solution for $E = \alpha$ also provided ψ_α is linearly independent with u

$$\varphi_\alpha(x) = \frac{u(x)}{W_c(x)} \tag{13}$$

where once again we have omitted an inessential constant factor.

The properties described above take place irrespective of any boundary value problem related to the differential equation (2). Here we shall consider boundary value problems of three kinds which are the most interesting from physical viewpoint:

(i) Regular Sturm-Liouville problem; the potential V_0 is bounded and continuous in $[a, b]$ which is a finite interval. We will consider only Dirichlet boundary conditions, $\psi_E(a) = 0$, $\psi_E(b) = 0$ imposed on smooth (infinitely differentiable) functions from $L^2(a, b)$ which form an initial domain of definition of h_0 which has a purely discrete spectrum.

(ii) Scattering potentials on a semiaxis, i.e. $V_0(x)$ is continuous and bounded from below for $x \in [0, \infty)$ and such that

$$\int_0^\infty x|V_0(x)|dx < \infty\,. \tag{14}$$

Here we impose on solutions to equation (2) the Dirichlet boundary condition at the origin only, $\psi_E(0) = 0$, which together with the condition of square integrability over the interval $[0, \infty)$ selects the bound states. The scattering states have an oscillating asymptotical behavior. The initial domain of definition of h_0 consists of infinitely differentiable functions vanishing for sufficiently large x and at the origin. The operator h_0 has a finite number of discrete levels and the continuous spectrum filling the positive semiaxis.

(iii) Confining and scattering potentials on the whole real line, $(a, b) = \mathbb{R}$. For confining potentials V_0 is locally bounded and $V_0(x) \to \infty$ when $|x| \to \infty$. Scattering potentials are selected by the condition

$$\int_{-\infty}^\infty |xV_0(x)|dx < \infty\,. \tag{15}$$

where $V_0(x)$ is continuous and semi-bounded from below. Operator h_0 is initially defined on the set of infinitely differentiable functions from $L^2(\mathbb{R})$ vanishing for sufficiently large $|x|$. For confining potentials the spectrum is purely discrete. The scattering potentials have a finite number of discrete levels and a two-fold degenerate continuous spectrum filling the positive semiaxis.

In all cases the operator h_0 is essentially self-adjoint and has a complete set of eigenfunctions (in the sense of generalized functions for continuous spectrum eigenfunctions) in corresponding Hilbert space (see e.g. [26]).

3 Complex SUSY partners of real potentials

3.1 General remarks

As we shall see both the analysis and the results strongly depend on the character of the initial Sturm-Liouville problem. Nevertheless, there is a property common to all eigenvalue problems which is essential for our analysis. We formulate it as the following

Proposition 1. *For a real potential $V_0(x)$ defining a self-adjoint operator h_0 in the space $L^2(a, b)$ any solution $\psi_\alpha(x)$ to the Schrödinger equation with $\mathrm{Im}\,(\alpha) \neq 0$ can vanish at no more then in one point of the interval $[a, b]$. For nonfinite values of a, b the statement should be understood in the sense of limit, i.e. for instance for $b = \infty$ this is $\lim_{x\to\infty} \psi_\alpha(x) = 0$. If $\mathrm{Im}\,(\alpha) = 0$ and exists $x_0 \in [a, b]$ such that $\psi_\alpha(x_0) = 0$ then up to an inessential constant factor $\psi_\alpha(x)$ is real $\forall x \in [a, b]$.*

Proof. The first part of the statement follows from the property of a self-adjoint operator to have only real eigenvalues. Indeed, if equations $\psi_\alpha(x_0) = \psi_\alpha(x_1) = 0$ took place for $\mathrm{Im}(\alpha) \neq 0$ and $x_0, x_1 \in [a, b]$, then the operator h_α, defined by the same differential expression h_0 and the zero boundary conditions at the ends of the interval $[x_0, x_1]$, being self adjoint would have the complex eigenvalue $E = \alpha$ which is impossible.

The second part follows from the property that for a real α the basis functions, $\psi_{1\alpha}(x)$ and $\psi_{2\alpha}(x)$ in the 2-dimensional space of solutions of equation (2) can always be chosen real so that any complex-valued solution $\psi_\alpha(x)$ is a linear combination $\psi_\alpha(x) = c_1\psi_{1\alpha}(x) + c_2\psi_{2\alpha}(x)$. For a finite value of x_0 from the equation $\psi_\alpha(x_0) = 0$ one of the constants, say c_2 (if $\psi_{2\alpha}(x_0) \neq 0$) can be found. Evidently, it is proportional to c_1 with a real proportionality coefficient. If $\psi_\alpha(\pm\infty) = 0$ the function $\psi_\alpha(x)$ is real-valued for a real α up to an inessential constant factor. The statement follows from a contradiction which appears if one supposes the opposite statement to be true. Indeed, if $\psi_\alpha(x)$ is a complex-valued function then $\psi^*_\alpha(x)$ (asterisk means complex conjugation) is linearly independent with $\psi_\alpha(x)$ and $\psi^*_\alpha(\pm\infty) = 0$ which is impossible. $\square$

We impose on solutions to equation (2) the same zero (Dirichlet) boundary conditions after 2-SUSY transformation. Thus we have two boundary value problems, initial and transformed, which we will denote (I) and (II) respectively.

For the usual (non-confluent) case, 2-SUSY transformation is reducible if 1-SUSY transformation with either $u = u_1$ or $u = u_2$ is 'good'. This is due to the fact that the chain can start from either $u = u_1$ or $u = u_2$. This contrasts with the confluent case since the chain starts now always from 1-SUSY transformation based on the transformation function u. Therefore any 2-SUSY transformation is irreducible if this transformation is 'bad'.

Below we analyze conditions for the transformation functions $u_{1,2}$ and factorization constants $\alpha_{1,2}$ giving (according to (6)) for a given real $V_0(x)$ a complex potential function $V_1(x)$ such that the operator h_1 defined in the corresponding Hilbert space according to the cases (i)-(iii) has a real spectrum coinciding with the spectrum of h_0 with the possible exception of one or two levels and the transformation is irreducible. We illustrate every possibility with the simplest example considering boundary value problems for $V_0(x) = 0$.

3.2 Regular Sturm-Liouville problem

The problem (I) is regular but the problem (II) may become singular if the potential $V_1(x)$ is unbounded in one of the bounds of the (finite) interval $[a, b]$ or in the both.

To distinguish irreducible second order transformations from reducible ones we have to start the analysis from first order transformations. From (4) it follows that if $\psi_E(a) = 0$ then $\widetilde{\psi}_E(a) = 0$ if and only if $u_1(a) = 0$. Hence to keep the zero boundary conditions we have to choose the function $u_1(x)$ vanishing both at $x = a$ and at $x = b$. This means that it is an eigenfunction of the problem (I). Since any eigenfunction except the ground state function has zeros inside the interval (a, b) we conclude that in this case there exists the only admissible first order SUSY transformation. It corresponds to $u_1(x) = \psi_0(x)$ (the ground state function) after which the ground state level is deleted. This means that any 2-SUSY transformation which does not involve the ground state function of the problem (I) is irreducible.

It is clear from (8-9) that if both u_1 (or equivalently u_2) and ψ_E, $E \neq \alpha_1, \alpha_2$, satisfy the zero boundary conditions then φ_E satisfies the zero boundary conditions also. Hence, to keep the zero boundary conditions after 2-SUSY transformation we have the following possible choices for u_1 and u_2:

(a) $u_1(a) = u_1(b) = u_2(a) = u_2(b) = 0$;

(b) $u_1(a) = u_1(b) = 0$, $u_2(a) = 0$ (or $u_2(b) = 0$), $u_2(b) \neq 0$ (or $u_2(a) \neq 0$);

(c) $u_1(a) = u_1(b) = 0$, $u_2(a) \neq 0$, $u_2(b) \neq 0$;

(d) $u_1(a) = u_2(b) = 0$ (or $u_1(b) = u_2(a) = 0$), $u_1(b) \neq 0$, $u_2(a) \neq 0$ (or $u_1(a) \neq 0$, $u_2(b) \neq 0$).

In case **(a)** both u_1 and u_2 are eigenfunctions of the problem (I) and there is no way to get a complex potential difference.

In case **(b)** u_1 is an eigenfunction of h_0, $u_1(x) = \psi_k(x)$ and $\alpha_1 = E_k$, $k = 0, 1, \ldots$. Hence, the level E_k is not present in the spectrum of h_1. The parameter α_2 can take any complex value except for $\alpha_2 = E_l$, $l = 1, 2 \ldots$, $l \neq k$ since in this case $u_2 = \psi_l$ and we are back in the conditions of the case (a). According to (10) the function φ_{α_2} satisfies the zero boundary conditions and, hence, the point $E = \alpha_2$ belongs to the discrete spectrum of h_1. Using Proposition 1 we conclude that in this case a complex potential V_1 is possible only for a complex value of α_2. The Hamiltonian h_1 has thus the complex discrete level $E = \alpha_2$. So, there are no ways to obtain a complex potential with a real spectrum in this case. The 2-SUSY transformation is reducible if $u_1(x) = \psi_0(x)$ (ground state function) and irreducible otherwise.

In case **(c)** u_1 is still an eigenfunction of h_0, $u_1(x) = \psi_k(x)$, $\alpha_1 = E_k$, $k = 0, 1, \ldots$ and the level E_k is not present in the spectrum of h_1 but there are no restrictions on u_2. Yet, the level $E = \alpha_2$ belongs to the spectrum of h_1 and if we want for the Hamiltonian h_1 to have a real spectrum we have to choose α_2 real. In this case complex potential differences can arise from formula (6) only if u_2 is a complex linear combination of two real linearly independent solutions of equation (2) with $h = h_0$. As it is shown in [14] if $\alpha_2 = E_l$ the Hamiltonian h_1 becomes non-diagonalizable. The second order transformation is irreducible provided $k > 0$. Another interesting feature we would like to mention is the possibility to get $\mathcal{PT}$-symmetric potentials by appropriate choice of the function $u_2(x)$ and if $V_0(x)$ has this property.

Example 1. For $x \in [-\pi, \pi]$ take $u_1 = \sin(n_0 x)$, $(\alpha_1 = n_0^2)$. $n_0 = 1, 2, \ldots$, $u_2 = \cos(ax + b)$ $(\alpha_2 = a^2)$, $a \in \mathbb{R}$, $a \neq n_0$, $\mathrm{Im}\,(b) \neq 0$. The potential V_1 is given by

$$V_1 = (n_0^2 - a^2)\,\frac{n_0^2[\cos(2ax + 2b) + 1] + a^2[\cos(2n_0 x) - 1]}{[n_0 \cos(n_0 x)\cos(ax + b) + a\sin(n_0 x)\sin(ax + b)]^2}\,. \tag{16}$$

If $a \neq \pm n/2$, $n = 1, 2, \ldots$ the spectrum of the Hamiltonian h_1 with potential (16) consists of all levels of $h_0 = -\partial_x^2$ which are $E_{n-1} = n^2/4$ except for $E = n_0^2$ and an additional level $E_a = a^2$. If $\mathrm{Re}\,(b) = 0$ this potential is explicitly $\mathcal{PT}$-symmetric.

One can find other examples of potentials one can get under these conditions in [14].

Consider finally the case **(d)**. Using Proposition 1 we conclude that if both α_1 and α_2 are real there is no way to obtain a complex potential difference. So, to be able to produce a complex potential V_1 we have to choose at least one of αs (say α_1) complex. In this case u_1 is nodeless and the intermediate Hamiltonian $\widetilde{h}$ is well-defined in $L^2(a, b)$ but its spectrum is completely different from the spectrum of h_0 since corresponding 1-SUSY transformation breaks the zero boundary condition at $x = b$. We thus can construct an irreducible SUSY model of a new type. Here also one can get $\mathcal{PT}$-symmetric potentials if $V_0(x)$ is $\mathcal{PT}$-symmetric and $\alpha_2 = \alpha_1^*$. This is readily seen for a symmetric interval $b = -a$. The last comment here is that according to (10) neither φ_{α_1} nor φ_{α_2} satisfy the zero boundary conditions. Therefore the Hamiltonian h_1 is strictly isospectral to h_0 and, hence, its spectrum is purely real.

Example 2. For $x \in [-\pi, \pi]$ taking $u_1 = \sin(a_1(x+\pi))$ $(\alpha_1 = a_1^2)$ and $u_2 = \sin(a_2(x-\pi))$, $(\alpha_2 = a_2^2)$, $a_1 \neq a_2$, $\mathrm{Im}\,(a_1^2) \neq 0$, $\mathrm{Im}\,(a_2^2) \neq 0$ one gets the following potential:

$$V_1 = (a_2^2 - a_1^2)\frac{a_2^2[1-\cos(2a_1(x+\pi))] - a_1^2[1-\cos(2a_2(x-\pi))]}{[a_1\cos(a_1(x+\pi))\sin(a_2(x-\pi)) - a_2\cos(a_2(x-\pi))\sin(a_1(x+\pi))]^2} \tag{17}$$

which is $\mathcal{PT}$-symmetric provided $a_2 = a_1^*$. The spectrum of the Hamiltonian h_1 coincides with the spectrum of h_0, $E_{n-1} = n^2/4$, $n = 1, 2, \ldots$. Although 1-SUSY transformations both with $u = u_1$ and $u = u_2$ produce potentials regular in the interval $(-\pi, \pi)$ they do not preserve the zero boundary conditions at both limiting points of (a, b) and therefore the 2-SUSY transformation is irreducible.

For the confluent case as it follows from (13) the function $u(x)$ should vanish at either bound of the interval $[a, b]$ and therefore it is one of the eigenfunctions of h_0, $u = \psi_k$ and $\alpha = E_k$, $k = 0, 1, \ldots$. Therefore to obtain a complex potential V_1 (6) one has to choose the constant c in (11) complex. This transformation is irreducible provided $u \neq \psi_0$. It keeps the spectrum unchanged since the function (13) satisfies the zero boundary conditions. In some cases (i.e. $V_0(-x) = V_0(x)$, $b = -a$, $x_0 = 0$ and $\mathrm{Re}(c) = 0$) the potential $V_1(x)$ is $\mathcal{PT}$-symmetric.

Example 3. For $x \in [-\pi, \pi]$ taking $x_0 = 0$ we get the potentials

$$V_1 = \mp n_0^2\,\frac{n_0(2c+x)\sin(n_0x) + 2\cos(n_0x) \pm 2}{[\sin(n_0x) \pm n_0(2c+x)]^2}\,. \tag{18}$$

Here the upper sign corresponds to $u = \cos(n_0x/2)$, $n_0 = 3, 5, 7, \ldots$ and the lower sign corresponds to $u = \sin(n_0x/2)$, $n_0 = 4, 6, 8, \ldots$, $(\alpha = n_0^2/4)$, $\mathrm{Im}\,(c) \neq 0)$. For $\mathrm{Re}(c) = 0$ these potentials are $\mathcal{PT}$-symmetric. The Hamiltonian h_1 is isospectral with $h_0 = -\partial_x^2$.

3.3 Scattering potentials on a semiaxis

As it was already mentioned at the beginning of Section 3.2, 1-SUSY transformation keeps unchanged the zero boundary condition (at the origin in the current case) if $u_1(0) = 0$, $u_1(x) \neq 0$ $\forall x \in (0, \infty)$. The zero boundary condition at the infinity for a transformed function is satisfied for any 1-SUSY transformation provided the transformation operator acts on a function vanishing at the infinity. Therefore any 2-SUSY transformation involving a transformation function vanishing at the origin and nodeless in the positive semiaxis is reducible.

Consider spectral problem (II). To keep the boundary condition at the origin after 2-SUSY transformation according to (8) one has to impose the same condition on one of the transformation functions, say $u_1(x)$, i.e. $u_1(0) = 0$. For the second order transformation to be irreducible one has to take care of presence of a positive node in $u_1(x)$. According to Proposition 1 if α_1 is complex the function $u_1(x)$ is nodeless in $(0, \infty)$ and 2-SUSY transformation is reducible. Therefore to construct an irreducible 2-SUSY transformation we have to choose only real values for α_1. According to the second part of the same proposition the function $u_1(x)$ can be chosen real without any loss of generality. So, we choose α_1 real, $u_1(x)$ real-valued and for $\alpha_2 \neq \alpha_1$ we enumerate the following possible choices for u_2:

(a) $u_2(0) = 0$;

(b) $u_2(0) \neq 0$ and $u_2(\infty) = 0$;

(c) $u_2(0) \neq 0$ and $u_2(\infty) = \infty$;

(d) $u_2(0) \neq 0$ and $u_2(x)$ has an oscillating asymptotics at the infinity.

For all cases (a)-(d) if α_1 is a point of the discrete spectrum of h_0, the function u_1 is a eigenfunction of the problem (I) (it is a bound state) and the point $E = \alpha_1$ does not belong to the discrete spectrum of the problem (II).

In case **(a)** by the same reason as it was explained above an irreducible 2-SUSY transformation is possible if α_2 is real but according to Proposition 1 it may produce only a real potential V_1. Nevertheless, one can get interesting complex potentials by reducible transformations with a complex α_2. If α_1 does not belong to the discrete spectrum of h_0 (i.e. $|u_1(\infty)| = \infty$) the 2-SUSY transformation is isospectral.

Example 4. Choose $u_1 = \sin(k_0 x)$ ($\alpha_1 = k_0^2 > 0$) and $u_2 = \sinh(ax)$ ($\alpha_2 = a^2 \in \mathbb{C}$, $\mathrm{Im}\alpha_2 \neq 0$). Formula (6) gives the potential

$$V_1 = (k_0^2 + a^2)\,\frac{k_0^2[\cosh(2ax) - 1] + a^2[\cos(2k_0 x) - 1]}{[k_0\cos(k_0 x)\sinh(ax) - a\sin(k_0 x)\cosh(ax)]^2}\,. \tag{19}$$

Formula (10) gives an oscillating solution of the Schrödinger equation with the potential (19) at $E = k_0^2$ but it is irregular at the origin. A regular at the origin solution at $E = k_0^2$ is unbounded as $x \to \infty$.

Similar to the case (a) in case **(b)** with a real α_2 according to Proposition 1 the potential V_1 remains real. Therefore to obtain a complex V_1 one has to choose α_2 complex. The function $\varphi_{\alpha_2}(x)$ (10) is vanishing at the origin but increasing at the infinity. So, the 2-SUSY transformation creates a complex potential with a real spectrum coinciding with the spectrum of h_0 with the possible exception of the point $E = \alpha_1$ (if it belongs to the spectrum of h_1). Here 2-SUSY transformation is reducible only if $u_1(x)$ is nodeless in $(0, \infty)$. According to Proposition 1 the function u_2 with a complex α_2 is nodeless and produces a "good" intermediate potential but the first order transformation operator based on u_2 does not transform eigenfunctions of h_0 into eigenfunctions of $\widetilde{h}$. So, the 2-SUSY transformation is irreducible if α_1 is real, the function $u_1(x)$ has a node in $(0, \infty)$ and α_2 is complex.

Example 5. Once again we choose $u_1 = \sin(k_0 x)$ ($\alpha_1 = k_0^2 > 0$) but $u_2 = e^{ax}$ ($\alpha_2 = -a^2$, $\mathrm{Re}(a) < 0$, $\mathrm{Im}\,(a^2) \neq 0$). The potential V_1 reads

$$V_1 = \frac{2k_0^2(k_0^2 + a^2)}{[k_0\cos(k_0 x) - a\sin(k_0 x)]^2}\,. \tag{20}$$

In contrast to the case (b) in case **(c)** function φ_{α_2} (10) is an eigenfunction of h_1 and $E = \alpha_2$ is its spectral point. Therefore to get Hamiltonian h_1 with a real spectrum we have to choose real values for α_2. Complex potentials can arise in this case if the function u_2 is a complex linear combination of two real linearly independent solutions to equation (2) at $E = \alpha_2$. So, the 2-SUSY transformation creates a new energy level $E = \alpha_2$. It is reducible if $u_1(x)$ is nodeless in $(0, \infty)$ and irreducible otherwise.

Example 6. Choice $u_1 = \sin(k_0 x)$ ($\alpha_1 = k_0^2$) and $u_2 = \cosh(ax + c)$ ($\alpha_2 = -a^2$), $k_0, a \in \mathbb{R}$ and $\mathrm{Im}(c) \neq 0$ results in the potential

$$V_1 = (k_0^2 + a^2)\,\frac{a^2[1 - \cos(2k_0 x)] + k_0^2[1 + \cosh(2ax + 2c)]}{[k_0\cos(k_0 x)\cosh(ax + c) - a\sin(k_0 x)\sinh(ax + c)]^2} \tag{21}$$

having a discrete level $E=-a^2$.

In case **(d)** $\alpha_2>0$. Therefore like in the previous case to have a complex V_1 u_2 should be a complex linear combination of two linearly independent solutions to equation (2). The function ψ_{α_2} belongs to the continuous spectrum of h_1 which has a purely real spectrum. In some cases indicated in [15] the point $E=\alpha_2$ is a spectral singularity of the Hamiltonian h_1.

Example 7. To illustrate this case we take $u_1=\sin(k_0x)$ ($\alpha_1=k_0^2$) and $u_2=\sin(k_1x+c)$ ($\alpha_2=k_1^2$), $k_0,k_1\in\mathbb{R}$, $k_0\neq k_1$, $\mathrm{Im}\,(c)\neq 0$ thus getting the potential

$$V_1=(k_1^2-k_0^2)\,\frac{k_1^2[1-\cos(2k_0x)]-k_0^2[1-\cos(2k_1+2c)]}{[k_0\cos(k_0x)\sin(k_1x+c)-k_1\sin(k_0x)\cos(k_1x+c)]^2}\,. \tag{22}$$

For the confluent transformation (6), (11), (13) to keep the zero boundary conditions we have to choose $u(0)=0$. Therefore by the same reason as it was explained above to get an irreducible 2-SUSY transformation we have to choose real values for α leading to a real-valued function $u(x)$. To obtain a complex potential V_1 one has to choose for the constant c from (11) a complex value. If $u(x)$ decreases at the infinity (i.e. it is an eigenfunction of h_1) the function $W_c(x)$ (11) is finite at the infinity and φ_α (13) is an eigenfunction of h_1. For x_0 one can choose both $x_0=0$ and $x_0=\infty$. Since $u(x)$ is square integrable different choices for x_0 affect only the value of c. If $u(x)$ increases at the infinity, one can choose $x_0=0$. Therefore the function $W_c(x)$ increases as $u^2(x)$ when $x\to\infty$ and the function φ_α (13) is an eigenfunction of h_1 too. If $\alpha>0$ the function $u(x)$ oscillates at the infinity as a linear combination $c_1\exp(-i\sqrt{\alpha}x)+c_2\exp(i\sqrt{\alpha}x)$. The integrand in (11) increases as a linear function of x and the potential V_1 keeping its oscillating behavior decreases like $1/x^2$. Therefore it does not satisfy condition (14) and it is not a scattering potential. This leads to the existence of a discrete level embedded into the continuous spectrum since the function φ_α (13) is vanishing at the origin and square integrable for $x\in[0,\infty)$. So, to get a complex potential by confluent transformation (6), (11) one has to choose α real, c complex and $x_0=0$. If α is not a discrete spectrum level then it appears as a new energy level for the Hamiltonian h_1. If $u(x)$ is nodeless in $(0,\infty)$ the transformation is reducible and irreducible otherwise.

Example 8. We choose $u=\sin(k_0x)$ ($\alpha=k_0^2>0$) and replace $c\to c/2$ ($\mathrm{Im}\,(c)\neq 0$). The potential

$$V_1=32k_0^2\sin(k_0x)\,\frac{\sin(k_0x)-k_0(x+c)\cos(k_0x)}{[\sin(2k_0x)-2k_0(x+c)]^2} \tag{23}$$

has the discrete level $E=k_0^2$ embedded into continuum of the scattering states.

3.4 Scattering and confining potentials on the whole real line

For a scattering potential the logarithmic derivative of any decreasing or increasing at $x\to\pm\infty$ solution $u(x)$ of the Schrödinger equation is asymptotically constant. For a confining potential the logarithmic derivative of any similar solution $u(x)$ is usually such that the product $\psi_E(x)u'(x)/u(x)$ increases or decreases together with $\psi_E(x)$ so that in the last case its asymptotics is square integrable; the behavior we will assume to take place. For a scattering potential if $u(x)$ has an oscillating asymptotics and $\psi_E(x)$ has an exponentially decreasing one the product $\psi_E(x)u'(x)/u(x)$ is exponentially decreasing also so that $\widetilde{L}\psi_E$ belongs to the discrete spectrum of $\widetilde{h}$ provided ψ_E belongs to the discrete spectrum of h_0. This means that if $u_1(x)$ is nodeless in $\mathbb{R}$ it produces a

“good” intermediate Hamiltonian $\widetilde{h}$. Thus, if α is real according to Proposition 1 any essentially complex-valued solution of equation (2) (i.e. a solution $u(x)$ such that $u'(x)/u(x)$ is a complex-valued function) is suitable for getting a “good” complex first order potential difference. Therefore if both α_1 and α_2 are real, any 2-SUSY transformation which may produce a complex potential V_1 is reducible. Let at least one of the factorization constants, say α_1 be complex (another constant, α_2, may be both real and complex). If $u_1(x)$ vanishes at one of the infinities then according to Proposition 1 it does not have real nodes and 2-SUSY transformation is reducible also. If $|u_1(x)| \to \infty$ when $|x| \to \infty$ then according to (10) the potential V_1 has a complex eigenvalue $E = \alpha_1$. We conclude, hence, that no irreducible 2-SUSY transformations giving a complex potential V_1 with a real spectrum exist. Of course this does not mean that such transformations cannot create complex potentials with a real spectrum but this means that any such a transformation can always be presented as a chain of two ‘good’ 1-SUSY transformations (with a possibility for the intermediate potential to be complex). In particular, if $\mathrm{Im}\,(\alpha_{1,2}) \neq 0$ and the functions $u_1(x)$, $u_2(x)$ vanish at different infinities (i.e. for a scattering potential they are two Jost solutions) one can get a complex potential isospectral with h_0 (hence, its spectrum is purely real). If for $V_0(x) = V_0(-x)$ in addition $\alpha_2 = \alpha_1^*$, $V_1(x)$ is $\mathcal{PT}$-symmetric. The last comment here is that irreducible 2-SUSY transformations can produce potentials with complex eigenvalues.

Example 9. Take $u_1 = \sinh(a_1(x - x_1))$ $(\alpha_1 = -a_1^2)$, $u_2 = \sinh(a_2(x - x_2))$ $(\alpha_2 = -a_2^2)$, $\mathrm{Im}\,(a_1^2) \neq 0$, $\mathrm{Im}\,(a_2^2) \neq 0$, $a_1 \neq a_2$, $x_1, x_2 \in \mathbb{R}$. The potential

$$V_1 = (a_2^2 - a_1^2)\,\frac{a_2^2[1 - \cosh(2a_1(x - x_1))] - a_1^2[1 - \cosh(2a_2(x - x_2)]}{[a_2 \cosh(a_2(x - x_2)) \sinh(a_1(x - x_1)) - a_1 \cosh(a_1(x - x_1)) \sinh(a_2(x - x_2))]^2} \tag{24}$$

has discrete levels at $E = -a_1^2$ and $E = -a_2^2$. If $x_2 = -x_1$ and $a_2 = a_1^*$ it is explicitly $\mathcal{PT}$-symmetric.

Consider finally the confluent case. If $\mathrm{Im}\,(\alpha) \neq 0$ and $u(x)$ increases at both infinities, as it follows from (13) the point $E = \alpha$ belongs to the spectrum of h_1. If $u(x)$ decreases at one of the infinities then according to Proposition 1 the function $u(x)$ has no real nodes and 2-SUSY transformation is reducible. Hence, irreducible 2-SUSY transformations creating complex potentials V_1 with a real spectrum are possible only with real values of α and $\mathrm{Im}\,(c) \neq 0$. In this case the function (13) has a decreasing asymptotical behavior both for increasing and decreasing $u(x)$ as well as for $u(x)$ having an oscillating asymptotical behavior. Therefore the level $E = \alpha$ belongs to the discrete spectrum of h_1. Simple potentials one can get in this way correspond to the choice of an eigenfunction (e.g. a bound state function) of h_0 as the transformation function $u(x)$ since in many cases it is described in terms of elementary functions. We do not illustrate these possibilities by examples and refer an interested reader to existing literature where this is done [19].

4 Concluding remarks and some perspectives

In summary, in this Letter a careful analysis aimed to distinguish irreducible transformations between all 2-SUSY transformations for three important Sturm-Liouville problems, namely, regular problem, problem on a half axis and problem on the whole real line is given. We remind that we call irreducible those second order transformations for which either (i) the intermediate Hamiltonian of corresponding chain of two transformations is not well defined in the same Hilbert space as

the initial and final Hamiltonians or (ii) the intermediate Hamiltonian is well defined but its eigenfunctions cannot be obtained by acting with the intertwining operator either on eigenfunctions of the initial Hamiltonian or on those of the final Hamiltonian. It is shown that for the whole real line the only possibility for such a transformation to be irreducible corresponds to the confluent case, i.e. to a chain of transformations with coinciding factorization constants. For problems on a half line and on a finite interval there are more possibilities. In particular, transformations of type (ii) lead to new irreducible SUSY models.

Using the property of SUSY transformations to provide us with a general solution of the Schrödinger equation at any fixed value of the energy one can observe in examples 4-7 an unusual property of SUSY transformations and an intriguing phenomenon concerning spectral properties of non-Hermitian operators. In the usual practice of SUSY transformations [13, 18, 24] if a transformation function corresponds to a spectral point (for the usual non-confluent case this may be only a point of the discrete spectrum) this point is deleted by the transformation. In examples 4-7 we used a transformation function corresponding to a point in the continuous part of the spectrum of the initial Hamiltonian but in contrast to conventional SUSY transformations (i.e. transformations between Hermitian operators) this point now still belongs to the continuous spectrum of the transformed problem. This statement follows from a general spectral theorem (see e.g. [27]) according to which the spectrum of a closed operator is a closed set and the property that if a point is removed from a closed interval of the real axis it is transformed into two (semi-)open subintervals. Of course, the operator h_1 with the initial domain of definition as described in Section 2 is not closed but it is closable and its closure coincides with h_1^{++} (see e.g. [28], theorem 6.3.2). Since the eigenfunctions of the operator h_1^+ coincide with complex conjugate eigenfunctions of h_1, the operator h_1^{++} has the same system of eigenfunctions as h_1. Moreover, the analysis of solutions of the Schrödinger equation at $E = k_0^2 > 0$ shows that a solution vanishing at the origin is unbounded. This contrasts with the usual quantum mechanical requirement that continuous spectrum eigenfunctions should be bounded. We think that instead of imposing the above quantum mechanical requirement on continuous spectrum eigenfunctions one should use the fact that these functions being ordinary locally integrable functions are in fact generalized eigenfunctions of the hamiltonian h_1 and should be considered as functionals over the domain of definition of h_1. (In other words one has to involve the notion of the Gelfand triplet into analysis [29].) From this point of view the question whether continuous spectrum eigenfunctions are bounded or not has no importance. Nevertheless, if necessary one can analyze the growth of generalized eigenfunctions for $x \to \infty$ (see theorem 6 in section 55 of [27]) as they are ordinary locally integrable functions.

We think that the results of the present Letter are important in view of the notion of $\mathcal{N}$-fold supersymmetry [30]. As far as this notion is applied to Hamiltonians of type (1) with supercharges built of differential intertwining operators acting in the same Hilbert space as the components of the super-Hamiltonian one can always apply a theorem [24] to factorize an Nth order in derivative intertwining operator to a superposition of first order operators thus replacing an Nth order transformation by a chain of only first order transformations. If all intermediate Hamiltonians of the chain are defined in the same Hilbert space as the initial and final Hamiltonians and at any step of transformations the eigenfunctions of two neighbor Hamiltonians are connected by corresponding transformation (i.e. intertwining) operator the $\mathcal{N}$-fold supersymmetry is reducible and, actually, all properties of the final Hamiltonian (as well as any intermediate Hamiltonian) can be understood at the level of a chain of simpler first order (i.e. usual) supersymmetry transfor-

mations. Evidently this is not the case if at least one Hamiltonian of the chain is not well-defined (case (i) of irreducible supersymmetry) or at least for one of the Hamiltonians we will not be able to get eigenfunctions by applying transformation operator to eigenfunctions of its SUSY partner (case (ii) of irreducible supersymmetry). The main result of the present Letter consists in formulating conditions on transformation functions to produce the simplest irreducible two-fold supersymmetry between real and complex potentials.

Another field of application of our results is the supersymmetric approach to the inverse scattering problem (so called supersymmetric inversion [31]). Since irreducible chains of transformations proved to be very efficient in supersymmetric inversion for usual Hermitian operators [32] we hope that our results open a way for wider applications of the method of supersymmetric inversion to complex potentials previously used for obtaining complex optical potentials only at the level of reducible transformations [33].

Acknowledgments

The work is partially supported by the grants SS-5103.2006.2 and RFBR-06-02-16719. The author is grateful to M.V. Ioffe for pointing out some references and to anonymous referee for useful comments.

References

[1] P.A.M. Dirac, Proc. Roy. Soc. London A 180 (1942) 1;
W. Pauli, Rev. Mod. Phys. 15 (1943) 175;
T. D. Lee, Phys. Rev. 95 (1954) 1329;
T. D. Lee and G.C. Wick, Nucl. Phys. B 9 (1969) 209.

[2] S.N. Gupta, Phys. Rev. 77 (1950) 294;
K. Bleuler, Helv. Phys. Acta. 23 (1950) 567.

[3] C.M. Bender and S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243.

[4] C.M. Bender, D.C. Brody and H.F. Jones, Phys. Rev. Lett. 89 (2002) 270401;
C.M. Bender, D.C. Brody and H.F. Jones, Phys. Rev. Lett. 92 (2004) 119902 (erratum).

[5] C.M. Bender, J.-H. Chen, and K.A. Milton, J. Phys. A: Math. Gen. 39 (2006) 1657.

[6] M. Znojil, Ed. Pseudo-Hermitian Hamiltonians in Quantum Physics (special issue of Czech. J. Phys. 55 (2005) No 9).

[7] F.G. Scholtz, H.B. Geyer and F.J.W. Hahne, Ann. Phys. 213 (1992) 74.

[8] L. Solombrino, J. Math. Phys. 43 (2002) 5439;
G. Scolarici and L. Solombrino, J. Math. Phys. 44 (2003) 4450.

[9] A. Mostafazadeh, J. Math. Phys. 43 (2002) 3944.

[10] A. Mostafazadeh, Preprint quant-ph/0603059.

[11] M. A. Naimark, Linear differential operators, Nauka, Moscow, 1969.

[12] M. Znojil, F. Cannata, B. Bagchi and R. Roychoudhury, Phys. Lett. B 483 (2000) 284; M. Znojil 2001 Czech. J. Phys. 51 (2001) 420.

[13] I. Aref'eva, D.J. Fernández, V. Hussin, J. Negro, L.M. Nieto, and B.F. Samsonov, eds., Progress in Supersym- metric Quantum Mechanics (special issue of J. Phys. A: Math. Gen. 37 (2004) No 43).

[14] B.F. Samsonov, J. Phys. A: Math. Gen. 38 (2005) L397.

[15] B.F. Samsonov, J. Phys. A: Math. Gen. 38 (2005) L571.

[16] A.A. Andrianov, F. Cannata, J.-P. Dedonder and M.V. Ioffe, Int. J. Mod. Phys. A 14 (1999) 2688.

[17] F. Cannata, G. Junker and J. Trost, Phys. Lett. A 246 (1998) 219.

[18] B. Mielnik and O. Rosas-Ortiz, J. Phys. A: Math. Gen. 43 (2004) 10007; D.J. Fernandez, AIP Conf. Proc. 744 (2005) 236.

[19] V. Milanović and Z. Ikonić, Phys. Lett. A 293 (2002) 29; J.S. Petrović, V. Milanović and Z. Ikonić, Phys. Lett. A 300 (2002) 595.

[20] A.A. Andrianov, M.V. Ioffe and V.P. Spiridonov, Phys. Lett. A 174 (1993) 273; A.A. Andrianov, F. Cannata, J.P. Dedonder and M.V. Ioffe, Int. J. Mod. Phys. A 10 (1995) 2683.

[21] B.F. Samsonov, Mod. Phys. Lett. 11 (1996) 1563; B.F. Samsonov, Phys. Lett. A 263 (1999) 274.

[22] A. A. Andrianov and F. Cannata, J. Phys. A: Math. Gen. A 37 (2004) 10297.

[23] B.F. Samsonov and A. Pupasov, Russ. Phys. J. 48(10), (2005) 20.

[24] V.G. Bagrov and B.F. Samsonov, Theor. Math. Phys. 104 (1995) 356; V.G. Bagrov and B.F. Samsonov, Phys. Part. Nucl 28 (1997) 374.

[25] M. Crum, Quart. J. Math. 6 (1955) 263; M.G. Krein, DAN SSSR (Doklady Akademii Nauk SSSR) 113 (1957) 970.

[26] F.A. Berezin M.A. and M.A. Shubin, The Schrödinger equation, Kluwer, Dordrecht, 1991.

[27] I.M. Glazman, Direct methods of qualitative spectral analysis of singular differential operators, Moscow, Fizmatgiz, 1963.

[28] A.I. Plesner, Spectral theory of linear operators, Nauka, Moscow, 1965.

[29] I.M. Gelfand and G.E. Shilov, Generalized functions, vol. 1-3, Academic, New York, 1964; I.M. Gelfand and N.Ya. Vilenkin, Generalized functions, vol. 4, Academic, New York, 1964.

[30] H. Aoyama, M. Sato and T. Tanaka, Nucl. Phys. B 619 (2001) 105;
H. Aoyama, N. Nakayama, M. Sato and T. Tanaka, Phys. Lett. B 519 (2001) 260;
T. Tanaka, Nucl. Phys. B 662 (2003) 413;
A. Gonzales-Lopes and T. Tanaka, J. Phys. A: Math. Gen. 39 (2006) 3715

[31] D. Baye, Phys. Rev. Lett. 58 (1987) 2738;
D. Baye and J.-M. Sparenberg, Phys. Rev. Lett. 73 (1994) 2789;
J.-M. Sparenberg and D. Baye, Phys. Rev. C 55 (1997) 2175;
J.-M. Sparenberg, D. Baye, and H. Leeb, Phys. Rev. C 61 (2000) 024605.
B. F. Samsonov and F. Stancu, B.F., Phys. Rev. C 66 (2002) 034001.

[32] B. F. Samsonov and F. Stancu, Phys. Rev. C 67 (2003) 054005;
J.-M. Sparenberg, B. F. Samsonov, F. Foucart and D. Baye, preprint quant-ph/0601101.

[33] J.-M. Sparenberg and D. Baye, Phys. Rev. C 54 (1996) 1309.